Near-Optimal Dimension Reduction for Facility Location

Lingxiao Huang [1]   Shaofeng H.-C. Jiang [2]   Robert Krauthgamer [3]   **Di Yue** [2]

August 29, 2024

[1]Nanjing University
[2]Peking University
[3]Weizmann Institute of Science

Dimension Reduction

Theorem (Johnson-Lindenstrauss lemma [Johnson-Lindenstrauss 84])

For all $n > 0$, $\varepsilon \in (0,1)$, there exists a random linear map $\pi \colon \mathbb{R}^d \to \mathbb{R}^m$, for $m = O(\varepsilon^{-2} \log n)$, such that for every $X \subset \mathbb{R}^d$ with $|X| = n$, with high probability

$$\forall x, y \in X, \|\pi(x) - \pi(y)\| \in (1 \pm \varepsilon) \|x - y\| .$$

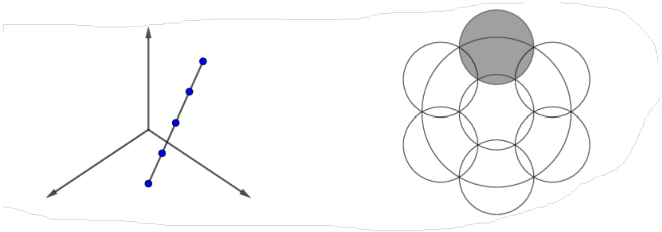
- JL mapping: $\pi \colon x \mapsto \frac{1}{\sqrt{m}} Gx$, where $G \in \mathbb{R}^{m \times d}$ with $g_{i,j} \sim i.i.d. N(0,1)$.
- $m = O(\varepsilon^{-2} \log n)$ is tight [Larsen, Nelson, FOCS 2017].
- Curse of dimensionality: target dimension $m = \Theta(\log n)$ is always too high to afford.
  - TSP does not admit a PTAS in dimension $m = \Theta(\log n)$ [Trevisan, SIAM J. Comput. 00].
  - Many problems have 2^{2^d} dependence in the (low-dimensional) PTAS.

Doubling Dimension

[Gupta, Krauthgamer, Lee, FOCS 03]

Solution: Use *intrinsic dimension*.

- $\mathrm{ddim}(X) :=$ the minimum $t \geq 0$, such that $\forall r > 0$, each ball in X of radius r can be covered by at most 2^t balls of radius $r/2$.



- Fundamental question: refine JL lemma such that m only depends on $\mathrm{ddim}(X)$.
- Preserve objective value for specific computational problems instead of pairwise distances. (i.e. $\mathrm{opt}(\pi(X)) \approx \mathrm{opt}(X)$)

Related Work on JL

- Preserve objective value for specific computational problems instead of pairwise distances. (i.e. $\mathrm{opt}(\pi(X)) \approx \mathrm{opt}(X)$)

Problems	**Approximation**	**Target Dimension**	**References**
Nearest Neighbor	$1+\varepsilon$	$O(\varepsilon^{-2}\mathrm{ddim})$	IN07
k-Center Clustering	$1+\varepsilon$	$O(\varepsilon^{-2}(\log k+\mathrm{ddim}))$	JKS24
k-Median / k-Means	$1+\varepsilon$	$O(\varepsilon^{-2}\log k)$	MMR19
Max-Cut	$1+\varepsilon$	$O(1/\varepsilon^{2})$	CJK23
MST	$1+\varepsilon$	$O(\varepsilon^{-2}\mathrm{ddim}\log\log n)$	NSIZ21
UFL	$O(1)$	$O(\varepsilon^{-2}\mathrm{ddim})$	NSIZ21
UFL	$1+\varepsilon$	$\tilde{O}(\varepsilon^{-2}\mathrm{ddim})$	This work

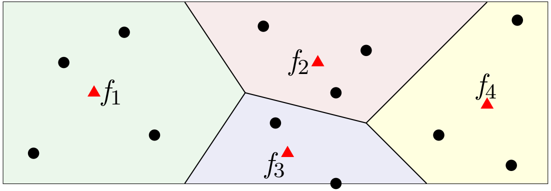
Uniform Facility Location (UFL)

Input: Point set $X \subset \mathbb{R}^d$ with bounded doubling dimension ddim, *opening cost* $\mathfrak{f} > 0$.
Goal: Find a set of *facilities* $F \subset \mathbb{R}^d$, so as to minimize the objective

$$\mathrm{cost}(X, F) := \underbrace{\mathfrak{f} \cdot |F|}_{\text{opening cost}} + \underbrace{\sum_{x \in X} \mathrm{dist}(x, F)}_{\text{connection cost}},$$

where $\mathrm{dist}(x, F) := \min_{y \in F} \mathrm{dist}(x, y)$ and $\mathrm{dist}(x, y) := \|x - y\|_2$.

- W.l.o.g., $\mathfrak{f} \equiv 1$ (by scaling the point set).
- Denote the *optimal value* by $\mathrm{ufl}(X)$.



Problem (Dimension reduction for UFL)

Given $\varepsilon, \delta > 0$, find target dimension $m = f(\mathrm{ddim}, \varepsilon, \delta)$, such that $\Pr[\mathrm{ufl}(\pi(X)) \in (1 \pm \varepsilon)\, \mathrm{ufl}(X)] \geq 1 - \delta$, where $\pi : \mathbb{R}^d \to \mathbb{R}^m$ is the JL mapping.

Results

Theorem (Dimension reduction for UFL)

$\Pr[\mathrm{ufl}(\pi(X)) \in (1 \pm \varepsilon)\,\mathrm{ufl}(X)] \geq 1 - \delta$, *for target dimension* $m := O(\varepsilon^{-2}\mathrm{ddim} \cdot \log(\delta^{-1}\varepsilon^{-1}\mathrm{ddim}))$.

Results

Theorem (Dimension reduction for UFL)

$\Pr[\mathrm{ufl}(\pi(X)) \in (1 \pm \varepsilon)\,\mathrm{ufl}(X)] \geq 1 - \delta$, *for target dimension* $m := O(\varepsilon^{-2}\mathrm{ddim} \cdot \log(\delta^{-1}\varepsilon^{-1}\mathrm{ddim}))$.

- Previous results: $O(1)$-approximation with $m = O(\varepsilon^{-2}\mathrm{ddim})$ [Narayanan, Silwal, Indyk, Zamir, ICML 21] or $(1 + \varepsilon)$-approximation with $m = O(\varepsilon^{-2} \log n)$ [Makarychev, Makarychev, Razenshteyn, STOC 19].

Results

Theorem (Dimension reduction for UFL)

$\Pr[\mathrm{ufl}(\pi(X)) \in (1 \pm \varepsilon)\,\mathrm{ufl}(X)] \geq 1 - \delta$, *for target dimension* $m := O(\varepsilon^{-2}\mathrm{ddim} \cdot \log(\delta^{-1}\varepsilon^{-1}\mathrm{ddim}))$.

- Previous results: $O(1)$-approximation with $m = O(\varepsilon^{-2}\mathrm{ddim})$ [Narayanan, Silwal, Indyk, Zamir, ICML 21] or $(1+\varepsilon)$-approximation with $m = O(\varepsilon^{-2}\log n)$ [Makarychev, Makarychev, Razenshteyn, STOC 19].

Theorem (PTAS for UFL)

There is an algorithm that computes a $(1+\varepsilon)$*-approximate solution for UFL, running in time* $(2^{m'}d + 2^{2^{m'}}) \cdot \tilde{O}(n)$*, for* $m' = O(\mathrm{ddim} \cdot \log(\mathrm{ddim}/\varepsilon))$.

Results

Theorem (Dimension reduction for UFL)

$\Pr[\mathrm{ufl}(\pi(X)) \in (1 \pm \varepsilon)\,\mathrm{ufl}(X)] \geq 1 - \delta$, *for target dimension* $m := O(\varepsilon^{-2}\mathrm{ddim} \cdot \log(\delta^{-1}\varepsilon^{-1}\mathrm{ddim}))$.

- Previous results: $O(1)$-approximation with $m = O(\varepsilon^{-2}\mathrm{ddim})$ [Narayanan, Silwal, Indyk, Zamir, ICML 21] or $(1+\varepsilon)$-approximation with $m = O(\varepsilon^{-2}\log n)$ [Makarychev, Makarychev, Razenshteyn, STOC 19].

Theorem (PTAS for UFL)

There is an algorithm that computes a $(1+\varepsilon)$*-approximate solution for UFL, running in time* $(2^{m'}d + 2^{2^{m'}}) \cdot \tilde{O}(n)$, *for* $m' = O(\mathrm{ddim} \cdot \log(\mathrm{ddim}/\varepsilon))$.

- Facilities are allowed to be picked from the (d-dimensional) ambient space.
- Previous result: Running in time $2^{2^{O(\mathrm{ddim}^2)}} d \cdot \tilde{O}(n)$, with facilities restricted to the same doubling metric [Cohen-Addad, Feldmann, Saulpic, JACM 21].

Proof Overview

- Upper bound $\mathrm{ufl}(\pi(X))$: $\mathrm{ufl}(\pi(X)) \leq \mathrm{cost}(\pi(X), \pi(F^*)) \lesssim \mathrm{cost}(X, F^*) = \mathrm{ufl}(X)$.

Proposition ([Johnson, Lindenstrauss 84] [Makarychev, Makarychev, Razenshteyn, STOC 19])

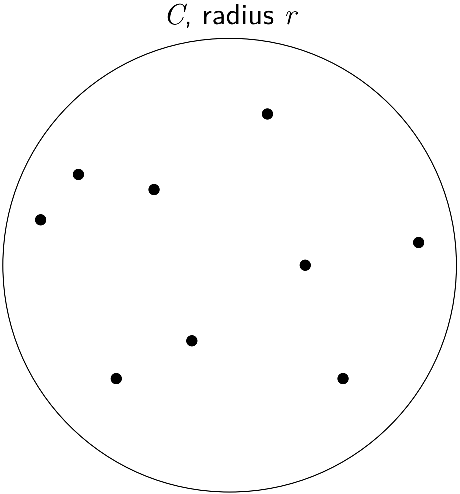
$\forall x, y \in \mathbb{R}^d$ *and* $t > 0$,

$$\mathbb{E}\left[\max\left\{0, \frac{\|\pi(x) - \pi(y)\|}{\|x - y\|} - (1 + t)\right\}\right] \leq \frac{1}{mt} e^{-t^2 m/2}.$$

Proof Overview

- Upper bound $\mathrm{ufl}(\pi(X))$: $\mathrm{ufl}(\pi(X)) \le \mathrm{cost}(\pi(X), \pi(F^*)) \lesssim \mathrm{cost}(X, F^*) = \mathrm{ufl}(X)$.
- Lower bound $\mathrm{ufl}(\pi(X))$
  (1) Partition X into "light" clusters $\mathbf{\Lambda} = \{C_1, C_2, \ldots, C_{|\mathbf{\Lambda}|}\}$, where $\mathrm{ufl}(C_i) = \Theta(\mathrm{ddim}/\varepsilon)^{O(\mathrm{ddim})}$.
  (2) Sub-additivity: $\mathrm{ufl}(X) \le \sum_{C \in \mathbf{\Lambda}} \mathrm{ufl}(C)$.
  (3) On each cluster $C \in \mathbf{\Lambda}$, $\mathrm{ufl}(C) \lesssim \mathrm{ufl}(\pi(C))$ in expectation.
  (4) $\sum_{C \in \mathbf{\Lambda}} \mathrm{ufl}(\pi(C)) \lesssim \mathrm{ufl}(\pi(X))$.

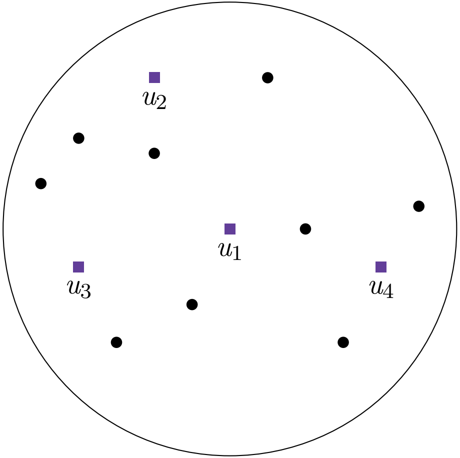
Step 1: Hierarchical Decomposition



Recursively partition a large cluster into small sub-clusters of half radius.

Step 1: Hierarchical Decomposition

C, radius r



Recursively partition a large cluster into small sub-clusters of half radius.

- Construct $(r/2)$-net N on C.

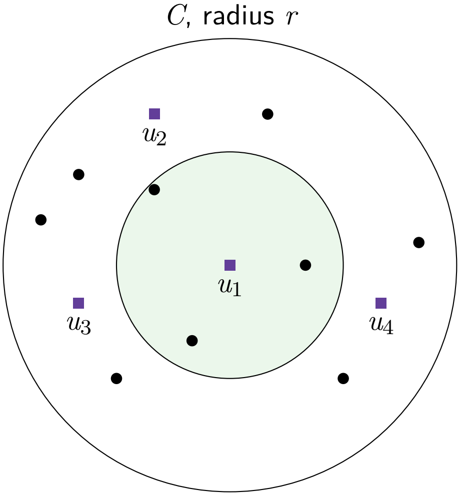
Definition (ρ-net)

- ρ-packing: $\forall x, y \in N$, $\mathrm{dist}(x, y) \geq \rho$.
- ρ-covering: $\forall x \in C$, $\exists y \in N$, s.t. $\mathrm{dist}(x, y) \leq \rho$.
- N is a ρ-net if it is both ρ-packing and ρ-covering for C.

Proposition (Packing property)

If S is ρ-packing then $|S| \leq (2\,\mathrm{diam}(S)/\rho)^{\mathrm{ddim}(S)}$.

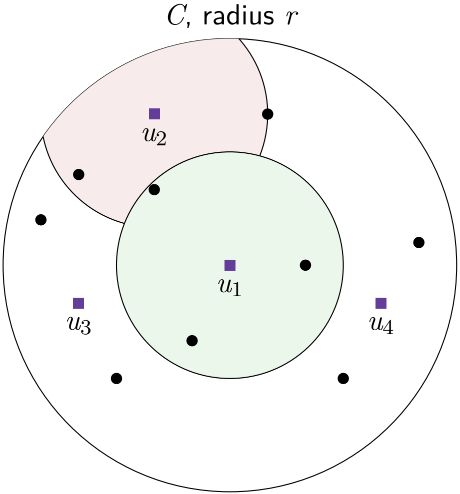
Step 1: Hierarchical Decomposition



Recursively partition a large cluster into small sub-clusters of half radius.

- Construct a level i child cluster $C_u \leftarrow C \cap B(u, r/2) \setminus \bigcup_{v \in N:\sigma(v)<\sigma(u)} B(v, r/2)$.

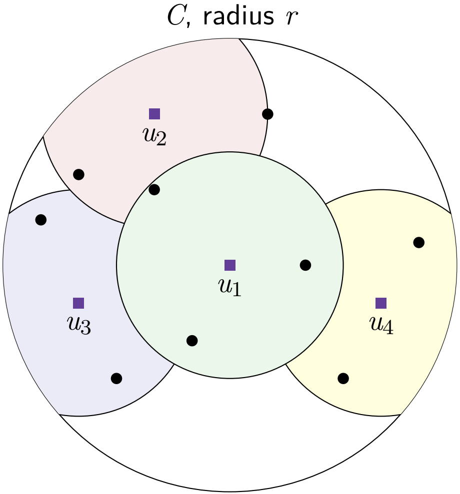
Step 1: Hierarchical Decomposition



Recursively partition a large cluster into small sub-clusters of half radius.

- Construct a level i child cluster $C_u \leftarrow C \cap B(u, r/2) \setminus \bigcup_{v \in N: \sigma(v) < \sigma(u)} B(v, r/2)$.

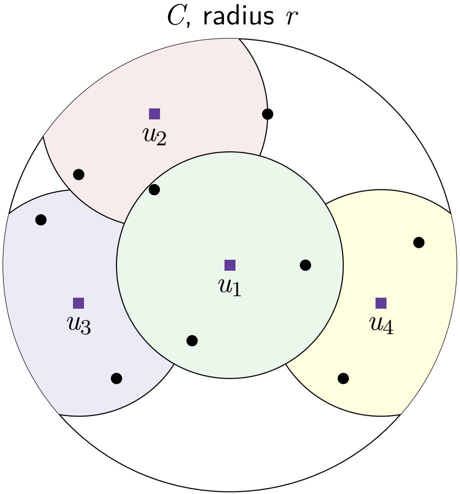
Step 1: Hierarchical Decomposition

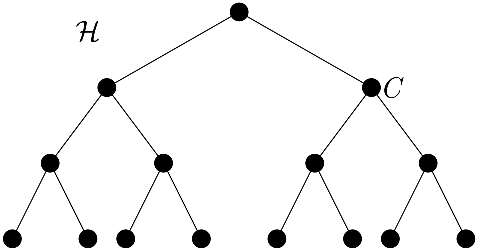


Recursively partition a large cluster into small sub-clusters of half radius.

- Construct a level i child cluster $C_u \leftarrow C \cap B(u, r/2) \setminus \bigcup_{v \in N : \sigma(v) < \sigma(u)} B(v, r/2)$.

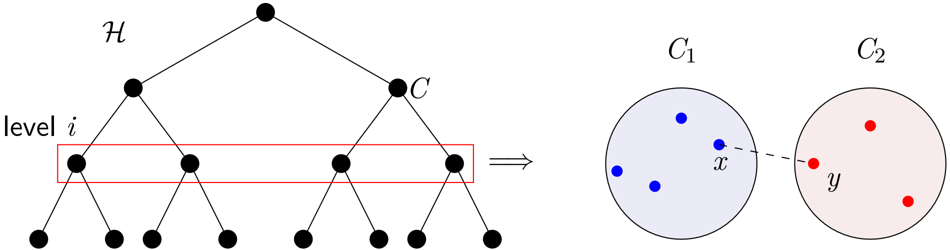
Step 1: Hierarchical Decomposition





- Node $\leftrightarrow$ cluster.
  - Root: X;
  - leaves: singletons;
  - level i: diameter $\Theta(2^i)$.
- Each node (cluster) has $2^{O(\mathrm{ddim})}$ child nodes (clusters).

Cut



- Cut [Talwar, STOC 04] [Cohen-Addad, Feldmann, Saulpic, JACM 21]: a pair of points (x, y) is cut by cluster C if $x \in C$ and $y \notin C$.
- Badly cut: (x, y) is *badly cut* if (x, y) is cut by some cluster C with $\mathrm{diam}(C) \geq \frac{\mathrm{ddim}}{\varepsilon^2} \cdot \|x - y\|$.
- There is a (random) hierarchical decomposition, such that $\forall x, y \in X$, $\Pr[(x, y)$ is badly cut$] \leq O(\varepsilon^2)$.

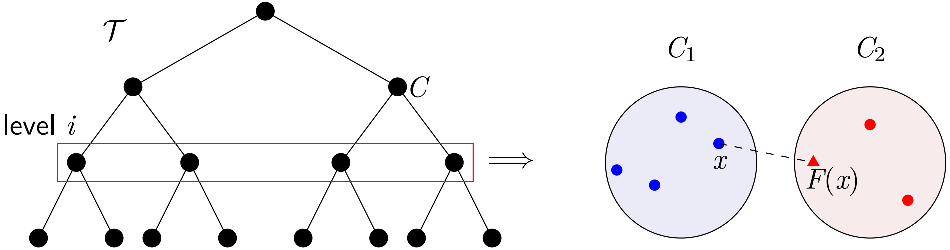
Step 2: Eliminating Badly-Cut Pairs

Fix facility set $F \subseteq X$.
Goal: $\forall x \in X$, if $(x, F(x))$ is cut by cluster C, then x and $F(x)$ are separated ($\|x - F(x)\| \gtrsim \mathrm{diam}(C)$).

Step 2: Eliminating Badly-Cut Pairs

Fix facility set $F \subseteq X$.
Goal: $\forall x \in X$, if $(x, F(x))$ is cut by cluster C, then x and $F(x)$ are separated ($\|x - F(x)\| \gtrsim \mathrm{diam}(C)$).

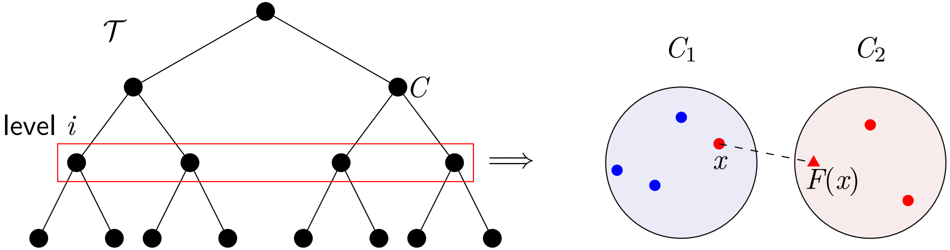
- For each level i:
  - For each $x \in X$, if $x \in C, F(x) \notin C$ and $\mathrm{diam}(C) \geq \frac{\mathrm{ddim}}{\varepsilon^2} \cdot \|x - F(x)\|$, then "move" x into the same level i cluster as $F(x)$.



Step 2: Eliminating Badly-Cut Pairs

Fix facility set $F \subseteq X$.
Goal: $\forall x \in X$, if $(x, F(x))$ is cut by cluster C, then x and $F(x)$ are separated ($\|x - F(x)\| \gtrsim \mathrm{diam}(C)$).

- For each level i:
    - For each $x \in X$, if $x \in C, F(x) \notin C$ and $\mathrm{diam}(C) \geq \frac{\mathrm{ddim}}{\varepsilon^2} \cdot \|x - F(x)\|$, then "move" x into the same level i cluster as $F(x)$.

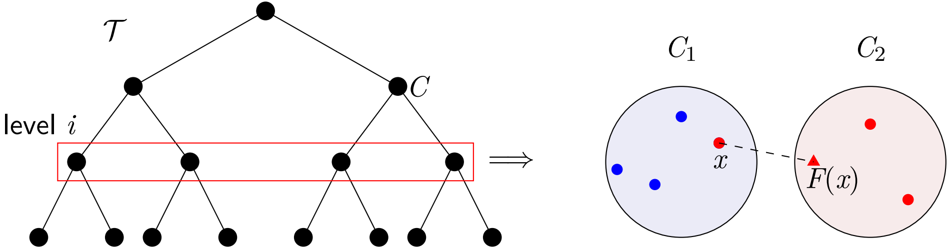


Step 2: Eliminating Badly-Cut Pairs

Fix facility set $F \subseteq X$.

Goal: $\forall x \in X$, if $(x, F(x))$ is cut by cluster C, then x and $F(x)$ are separated ($\|x - F(x)\| \gtrsim \mathrm{diam}(C)$).

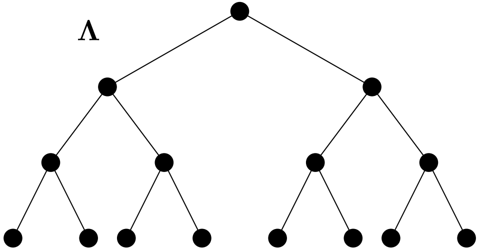
- For each level i:
  - For each $x \in X$, if $x \in C, F(x) \notin C$ and $\mathrm{diam}(C) \geq \frac{\mathrm{ddim}}{\varepsilon^2} \cdot \|x - F(x)\|$, then "move" x into the same level i cluster as $F(x)$.



- On the new decomposition $\mathcal{T}$, $(x, F(x))$ is not badly-cut.
  - Separation: If $(x, F(x))$ is cut by cluster C, then $\|x - F(x)\| \geq \frac{\varepsilon^2}{\mathrm{ddim}} \mathrm{diam}(C)$.

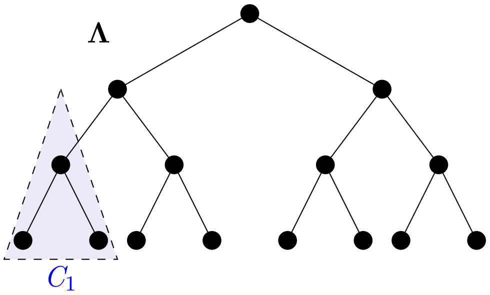
Step 3: Partition

Given threshold $\kappa = \Theta(\mathrm{ddim}/\varepsilon)^{O(\mathrm{ddim})}$, find the lowest level "heavy cluster" ($\mathrm{ufl}(C) \geq \kappa$) in a bottom-up manner.



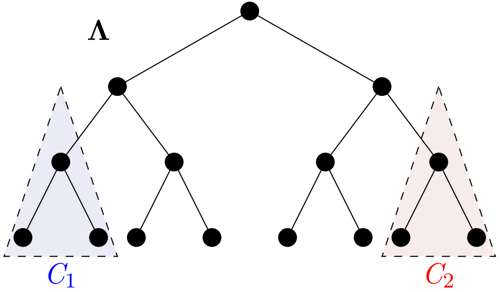
Step 3: Partition

Given threshold $\kappa = \Theta(\mathrm{ddim}/\varepsilon)^{O(\mathrm{ddim})}$, find the lowest level "heavy cluster" ($\mathrm{ufl}(C) \geq \kappa$) in a bottom-up manner.



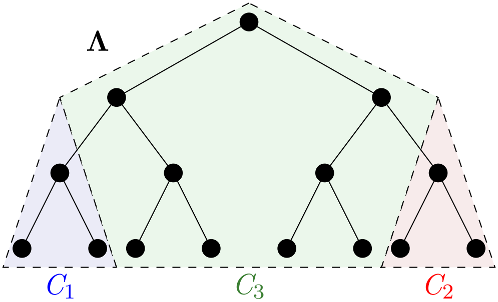
Step 3: Partition

Given threshold $\kappa = \Theta(\mathrm{ddim}/\varepsilon)^{O(\mathrm{ddim})}$, find the lowest level "heavy cluster" ($\mathrm{ufl}(C) \geq \kappa$) in a bottom-up manner.



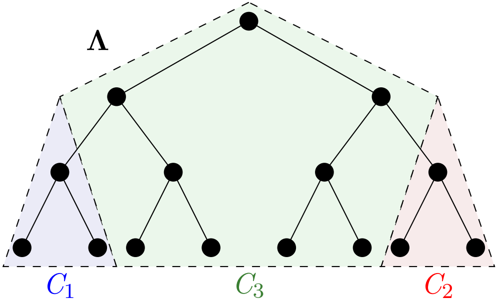
Step 3: Partition

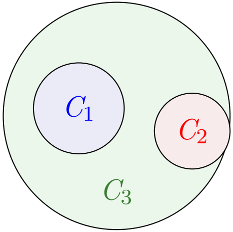
Given threshold $\kappa = \Theta(\mathrm{ddim}/\varepsilon)^{O(\mathrm{ddim})}$, find the lowest level "heavy cluster" ($\mathrm{ufl}(C) \geq \kappa$) in a bottom-up manner.



Step 3: Partition

Given threshold $\kappa = \Theta(\mathrm{ddim}/\varepsilon)^{O(\mathrm{ddim})}$, find the lowest level "heavy cluster" ($\mathrm{ufl}(C) \geq \kappa$) in a bottom-up manner.

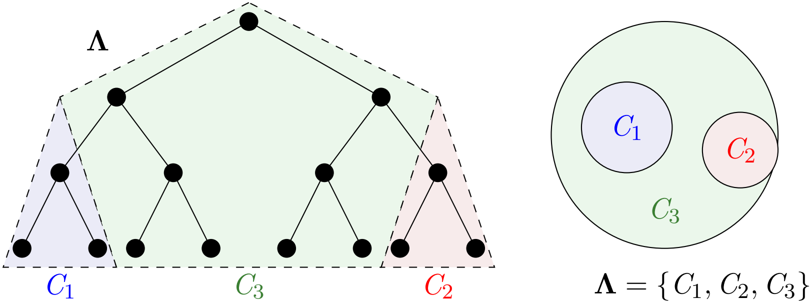




$\mathbf{\Lambda} = \{C_1, C_2, C_3\}$

Step 3: Partition

Given threshold $\kappa = \Theta(\mathrm{ddim}/\varepsilon)^{O(\mathrm{ddim})}$, find the lowest level "heavy cluster" ($\mathrm{ufl}(C) \geq \kappa$) in a bottom-up manner.



- Bounded optimal value: $\forall C \in \mathbf{\Lambda}$, $\kappa \leq \mathrm{ufl}(C) \leq 2^{O(\mathrm{ddim})}\kappa$.

JL on Each Cluster

- Bounded optimal value: $\forall C \in \mathbf{\Lambda}$, $\kappa \leq \mathrm{ufl}(C) \leq 2^{O(\mathrm{ddim})}\kappa$.
- Equivalent to τ-median on C, where $\tau = 2^{O(\mathrm{ddim})}\kappa = \Theta(\mathrm{ddim}/\varepsilon)^{O(\mathrm{ddim})}$.

Lemma (Refined from [Makarychev, Makarychev, Razenshteyn, STOC 19])

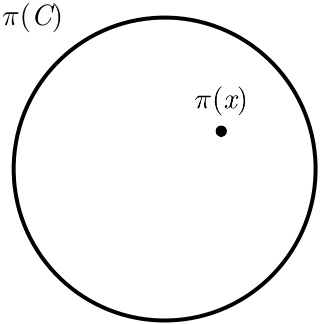
Assume $\mathrm{ufl}(C) \leq \tau$. *Then for* $m = \Omega(\varepsilon^{-2}\log(1/\varepsilon))$,

$$\Pr\left[\mathrm{ufl}(\pi(C)) \leq \frac{1}{1+\varepsilon}\mathrm{ufl}(C)\right] \leq \tau^3 \cdot e^{-\Omega(\varepsilon^2 m)}.$$

- Choose target dimension $m = O(\varepsilon^{-2}\log\tau) = O(\varepsilon^{-2}\mathrm{ddim} \cdot \log(\mathrm{ddim}/\varepsilon))$.
- On each cluster $C \in \mathbf{\Lambda}$, $\mathrm{ufl}(C) \lesssim \mathrm{ufl}(\pi(C))$ in expectation.

Lower Bound $\mathrm{ufl}(\pi(X))$

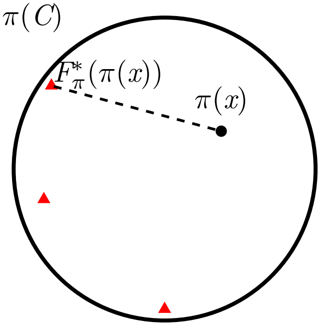
- Goal: $\sum_{C \in \mathbf{\Lambda}} \mathrm{ufl}(\pi(C)) \lesssim \mathrm{ufl}(\pi(X))$.
- Idea: Open a "local" facility set $F'_{\pi(C)}$ for each $\pi(C)$ and show $\mathrm{dist}(\pi(x), F'_{\pi(C)}) \lesssim \mathrm{dist}(\pi(x), F^*_\pi), \forall x \in C$.



$F'_{\pi(C)} :=$

Lower Bound $\mathrm{ufl}(\pi(X))$

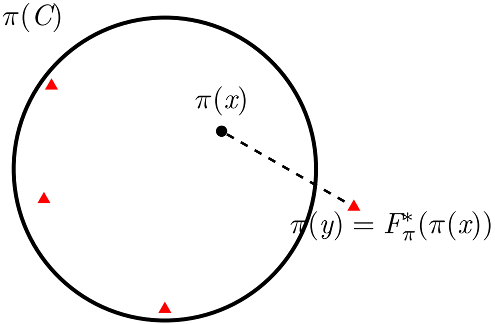
- Goal: $\sum_{C\in\mathbf{\Lambda}} \mathrm{ufl}(\pi(C)) \lesssim \mathrm{ufl}(\pi(X))$.
- Idea: Open a "local" facility set $F'_{\pi(C)}$ for each $\pi(C)$ and show $\mathrm{dist}(\pi(x), F'_{\pi(C)}) \lesssim \mathrm{dist}(\pi(x), F^*_\pi), \forall x \in C$.



$$F'_{\pi(C)} := (F^*_\pi \cap \pi(C))$$

Lower Bound $\mathrm{ufl}(\pi(X))$

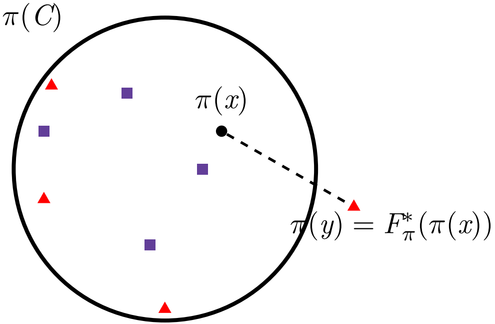
- Goal: $\sum_{C \in \mathbf{\Lambda}} \mathrm{ufl}(\pi(C)) \lesssim \mathrm{ufl}(\pi(X))$.
- Idea: Open a "local" facility set $F'_{\pi(C)}$ for each $\pi(C)$ and show $\mathrm{dist}(\pi(x), F'_{\pi(C)}) \lesssim \mathrm{dist}(\pi(x), F^*_\pi), \forall x \in C$.



$$F'_{\pi(C)} := (F^*_\pi \cap \pi(C))$$

Lower Bound $\mathrm{ufl}(\pi(X))$

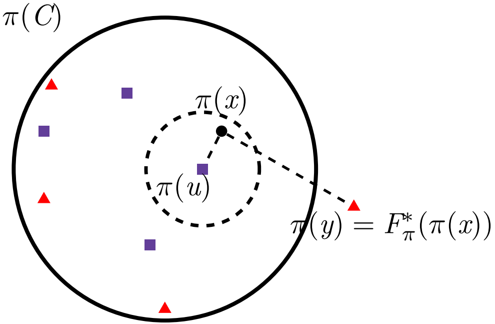
- Goal: $\sum_{C \in \mathbf{\Lambda}} \mathrm{ufl}(\pi(C)) \lesssim \mathrm{ufl}(\pi(X))$.
- Idea: Open a "local" facility set $F'_{\pi(C)}$ for each $\pi(C)$ and show $\mathrm{dist}(\pi(x), F'_{\pi(C)}) \lesssim \mathrm{dist}(\pi(x), F^*_\pi), \forall x \in C$.



$$F'_{\pi(C)} := (F^*_\pi \cap \pi(C)) \cup \pi(\overbrace{N_C}^{\sigma \cdot \mathrm{diam}(C)\text{-net on } C}).$$

Lower Bound $\mathrm{ufl}(\pi(X))$

- Goal: $\sum_{C\in\Lambda} \mathrm{ufl}(\pi(C)) \lesssim \mathrm{ufl}(\pi(X))$.
- Idea: Open a "local" facility set $F'_{\pi(C)}$ for each $\pi(C)$ and show $\mathrm{dist}(\pi(x), F'_{\pi(C)}) \lesssim \mathrm{dist}(\pi(x), F^*_\pi), \forall x \in C$.

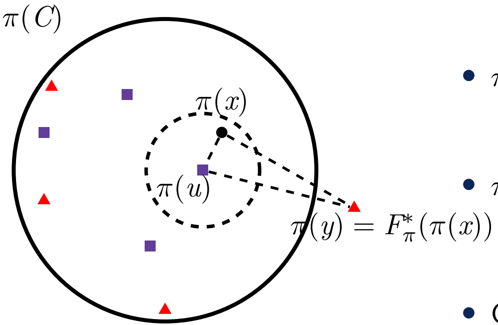


$$F'_{\pi(C)} := (F^*_\pi \cap \pi(C)) \cup \pi(\overbrace{N_C}^{\sigma\cdot\mathrm{diam}(C)\text{-net on } C}).$$

Lower Bound $\mathrm{ufl}(\pi(X))$

- Goal: $\sum_{C \in \Lambda} \mathrm{ufl}(\pi(C)) \lesssim \mathrm{ufl}(\pi(X))$.
- Idea: Open a "local" facility set $F'_{\pi(C)}$ for each $\pi(C)$ and show $\mathrm{dist}(\pi(x), F'_{\pi(C)}) \lesssim \mathrm{dist}(\pi(x), F^*_\pi), \forall x \in C$.

$$F'_{\pi(C)} := (F^*_\pi \cap \pi(C)) \cup \pi(\overbrace{N_C}^{\sigma \cdot \mathrm{diam}(C)\text{-net on } C}).$$

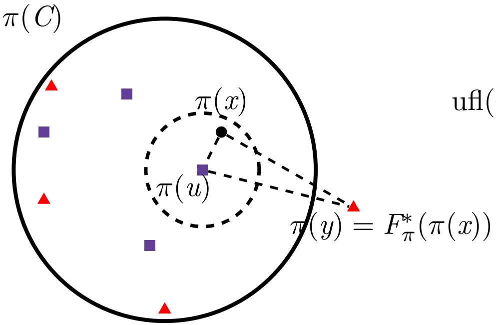


- $\pi(x)$ and $\pi(u)$ are close.
  - $\|x - u\| \le \sigma \cdot \mathrm{diam}(C) \implies \|\pi(x) - \pi(u)\| \le O(\sigma) \cdot \mathrm{diam}(C)$ w.h.p.
- $\pi(x)$ and $\pi(y)$ are separated.
  - (x, y) is not badly cut $\implies \|x - y\| \ge \frac{\varepsilon^2}{\mathrm{ddim}} \cdot \mathrm{diam}(C)$.
  - $\|\pi(x) - \pi(y)\| \ge \frac{\Omega(\varepsilon^2)}{\mathrm{ddim}} \cdot \mathrm{diam}(C)$ w.h.p.
- Conclusion: $\|\pi(x) - \pi(u)\| \le \|\pi(x) - \pi(y)\|$.

Lower Bound $\mathrm{ufl}(\pi(X))$

- Goal: $\sum_{C\in\mathbf{\Lambda}} \mathrm{ufl}(\pi(C)) \lesssim \mathrm{ufl}(\pi(X))$.
- Idea: Open a "local" facility set $F'_{\pi(C)}$ for each $\pi(C)$ and show $\mathrm{dist}(\pi(x), F'_{\pi(C)}) \lesssim \mathrm{dist}(\pi(x), F^*_\pi), \forall x \in C$.

$$F'_{\pi(C)} := (F^*_\pi \cap \pi(C)) \cup \pi(\overbrace{N_C}^{\sigma\cdot\mathrm{diam}(C)\text{-net on } C}).$$



$$\mathrm{ufl}(\pi(C)) \le \sum_{x\in C} \mathrm{dist}(\pi(x), F'_{\pi(C)}) + \left|F'_{\pi(C)}\right|$$

$$\le \underbrace{\sum_{x\in C} \mathrm{dist}(\pi(x), F^*_\pi)}_{\text{connection cost on } \pi(C)} + \underbrace{|F^*_\pi \cap \pi(C)|}_{\text{opening cost on } \pi(C)} + |N_C|.$$

$$\Longrightarrow \sum_{C\in\mathbf{\Lambda}} \mathrm{ufl}(\pi(C)) \le \mathrm{ufl}(\pi(X)) + \text{err}.$$

PTAS for UFL

- Construct the partition $\mathbf{\Lambda}$.
  - Hierarchical decomposition $\mathcal{H}$.
  - Eliminate badly-cut pairs $(x, F(x)) \Longrightarrow \mathcal{T}$.
  - Construct $\mathbf{\Lambda}$ via $\mathcal{T}$.
- Construct near-optimal clustering $\mathcal{C}$.
  - For each $C \in \mathbf{\Lambda}$, construct the near optimal clustering $\mathcal{C}_C$ for $\pi(C)$.
  - $\mathcal{C} := \bigcup_{C \in \mathbf{\Lambda}} \mathcal{C}_C$.
- Construct facility set F.
  - For each $X_i \in \mathcal{C}$, compute the (approximate) 1-median center f_i for X_i.
  - $F := \{f_1, f_2, \ldots, f_s\}$.

Thank you!